Identification, Reduction, and Refinement of Model Parameters by the Eigensystem Realization Algorithm

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A new identification scheme is developed to determine model parameters of vibrating structures from experimental data. The eigensystem realization algorithm, which forms the basis for a rational choice of model size determined by the singular values and accuracy indicators, is exploited and extended here to identify the mass, damping, and stiffness matrices directly from the singular value decomposition of the Hankel matrix. This approach provides a closed-loop identification framework that feeds back the output of eigensystem realization algorithm (ERA) and modifies the elements and the dimensions of the matrices accordingly by incorporating the model reduction and model refinement algorithm developed in this paper.

Introduction

OVER the last several years, the investigation of realization theory in the area of modal analysis was particularly fruitful; much useful work was accomplished.[1-5] Realization theory stems from control engineering, which primarily deals with the formulation of a state-space representation from a given transfer function. Under the interaction of structure and control disciplines, researchers have found its important role in the identification of modal parameters for flexible structures. Though the task of modal-parameter identification is treated in several ways by different researchers,[6] most of the existing methods can be reformulated in a unified way under the framework of system realization theory. Among the various methods, eigensystem realization algorithm (ERA)[1] is a special form of realization, the so-called internal balanced realization[7]; Prony's algorithm,[8] Ibrahim time-domain method,[9] and the polyreference technique[10] all are related to a canonical-form realization technique[2,5]; least-square regression technique is merely a nonminimum realization of the dynamic system under test.[2] Because of this close correlation, it is hoped that system realization theory provides the unification of the many possible approaches for modal-parameter identificaion and paves the way for the discovery of new identification techniques.

To design controls for large space structures, it is necessary to have a mathematical model that will adequately describe the system's motion. It is the purpose of system realization to construct a state-space model from experimental data. Among the many models realized that have the same input-output relations, the one that possesses the minimum order is particularly important, whereas among the various methods for constructing a minimum-order model, the realization technique using singular value decomposition (SVD) on a Hankel matrix is comparatively suitable for the purpose of modal-parameter identification. Along this line, the ERA is developed and applied successfully to several sets of structural dynamics data.[1,4] The main advantages of the ERA include 1) the SVD algorithm is simple and numerically stable, 2) data from more than one test can be used simultaneously to efficiently identify closely spaced eigenvalues, 3) the number of nonzero singular values serves as an accurate criterion for rank detection. In the presence of noise, accuracy indicators[2] also have been developed to quantitatively identify the system and noise modes.

Although identification of modal parameters such as damping, frequencies, mode shapes, and modal-participation factors using ERA has been discussed widely in the literature, formal direct application of ERA to the identification of model parameters such as mass, damping, and stiffness matrices (M,C,K) for flexible structures has not been addressed. The purpose of this paper is to apply ERA to derive a novel identification technique for M, C, and K. A common feature for the existing methods is that the dimension of the identified matrices M, C, and K is always equal to the number of spatial points at which dynamic response is recorded. In the literature, several identification schemes[11-15] have been reported to find a set of M, C, and K of "assumed dimension" from experimental data. However, it is plainly difficult to justify this assumed dimension as the best one among all the possible candidates, since the number of measurement points may not in any sense be responsible for the rank of the Hankel matrix, which is an indication of the number of modes existing in the response of the structure under testing. If we regard the assumed dimension of the matrices as the input of an identification loop and the resulting M, C, and K as the corresponding output, then the conventional approaches are obviously an open-loop identification scheme since the output of the loop does not serve as a corrective quantity for the input. To close this identification loop, a reasonable approach is to feed back the results of ERA, by noting that the ERA method forms the basis for a rational choice of model size determined by the singular values and accuracy indicators, and to modify the identified matrices accordingly.

In this paper, we will derive explicit expressions for M, C, and K using the SVD technique. Recursive formulas of model reduction or model refinement for M, C, and K, depending on the relations between the assumed dimensions of the matrices and the output of the ERA, are obtained as well.

Identification of M, C, and K using ERA

A continuous dynamic system with infinite degrees of freedom, after neglecting its high-frequency response, can be

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approximated by the following linear matrix differential equation:

$$M\ddot{x} + C\dot{x} + Kx = u \tag{1}$$

where x is the $n \times 1$ configuration vector of physical displacements, u the $n \times 1$ force vector, M the $n \times n$ positive-definite mass matrix, C the $n \times n$ positive-semidefinite damping matrix, and K the $n \times n$ positive-semidefinite stiffness matrix. Dots denote differentiation with respect to time. The problem in which we are interested is to determine the matrices M,C,K such that the dynamic response of the system is best fitted by Eq. (1). Taking the Laplace transform of Eq. (1) and assuming zero initial conditions gives

$$B(s)X(s) = U(s) \tag{2}$$

where

$$B(s) = Ms^2 + Cs + K \tag{3}$$

Here, s is the Laplace variable, $U(s)$ the applied force vector, and $X(s)$ the resulting displacement vector in the Laplace domain. $B(s)$ is called the system matrix, and the associated transfer function matrix $\Phi(s)$ is defined as

$$\Phi(s) = B(s)^{-1} \tag{4}$$

The Markov parameters ϕ_i of the system are related to $\Phi(s)$ via the expansion

$$\Phi(s) = \phi_0 + \frac{\phi_1}{s} + \frac{\phi_2}{s^2} + \frac{\phi_3}{s^3} + \cdots \tag{5}$$

Some simple identification algorithms have been derived using $B(s)$ or $\Phi(s)$. From Eq. (3) and Eq. (4) we have

$$K = B(s)\big|_{s=0} = \Phi(s)^{-1}\big|_{s=0} \tag{6a}$$

$$C = \frac{\mathrm{d}B(s)}{\mathrm{d}s}\Big|_{s=0} = -K\left[\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\Big|_{s=0}\right]K \tag{6b}$$

$$M = \frac{\mathrm{d}^2B(s)}{\mathrm{d}s^2} = -K\left[\frac{\mathrm{d}\Phi(s)}{\mathrm{d}s}\Big|_{s=0}\right]C - \tfrac{1}{2}K\left[\frac{\mathrm{d}^2\Phi(s)}{\mathrm{d}s^2}\Big|_{s=0}\right]K \tag{6c}$$

Using the preceding formulas and the measurement of one row or one column of $\Phi(s)$, M, C, and K can be identified.[12] On the other hand, if Markov parameters are considered, we can derive expressions for M, C, and K using Eqs. (3–5) as

$$M = \phi_2^{-1} \tag{7a}$$

$$C = -\phi_2^{-1}\phi_3\phi_2^{-1} \tag{7b}$$

$$K = \phi_2^{-1}(\phi_3\phi_2^{-1})^2 - \phi_2^{-1}\phi_4\phi_2^{-1} \tag{7c}$$

The Markov parameters ϕ_i can be evaluated using impulse response data.[1,11] The preceding methods are simple in their mathematic formulation, but difficulties may be encountered for practical use. First, these algorithms are vulnerable to rapid deterioration of identification accuracy in the presence of noisy data; second, the identified mass, damping, and stiffness matrices are of dimension $n \times n$, while recalling n is the number of measurement points that may not, in any sense, completely reflect the true order of the dynamic system under testing. To circumvent these difficulties, we need an identification technique that is robust against noisy data and has a rank detection ability. With the help of ERA, such an identification technique can be constructed.

It is better at this moment to review some properties of ERA. We first rewrite Eq. (1) in a state-space representation:

$$\dot{X} = FX + Gu \tag{8a}$$

$$Z = HX \tag{8b}$$

where

$$X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

and

$$F = \begin{pmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}K & -M^{-1}C \end{pmatrix}$$

$$G = (0_{n\times n} \quad M^{-1})^T$$

$$H = (I_{n\times n} \quad 0_{n\times n})$$

The transfer function from u to Z is found immediately as

$$\frac{Z(s)}{U(s)} = H(sI-F)^{-1}G \tag{9}$$

Using the definition of F, G, and H, it is straightforward to see

$$\Phi(s) = (Ms^2 + Cs + K)^{-1} = H(sI-F)^{-1}G$$

If we expand $H(sI-F)^{-1}G$ as a power series of s

$$H(sI-F)^{-1}G = \frac{HG}{s} + \frac{HFG}{s^2} + \frac{HF^2G}{s^3} + \cdots \tag{10}$$

and compare the corresponding coefficients of s in Eq. (5) and Eq. (10), we then have

$$\phi_i = HF^{i-1}G, \qquad i = 1,2,\cdots \tag{11}$$

Especially,

$$HG = \phi_1 = 0 \tag{12a}$$

$$HFG = \phi_2 = M^{-1} \tag{12b}$$

Equation (12a) results because the elements of $B(s)$ are quadratic functions of s, and we must have $\phi_0 = \phi_1 = 0$ in the expansion of $\Phi(s)$. Equation (12b) is the same as Eq. (7a). The problem of system realization is that given the Markov parameter ϕ_i obtained from impulse response data, constant matrices (F,G,H) are constructed in terms of ϕ_i such that the identities of Eq. (11) hold. The algorithm begins by forming the $2n \times 2n$ block Hankel matrix

$$T(n,k) = \begin{bmatrix} \phi_{k+1} & \phi_{k+2} & \cdots & \phi_{k+2n} \\ \phi_{k+2} & \phi_{k+3} & \cdots & \phi_{k+2n+1} \\ \phi_{k+2n} & \phi_{k+2n+1} & \cdots & \phi_{k+4n-1} \end{bmatrix} \tag{13}$$

By using the identities in Eq. (11), we have the following useful relation:

$$T(n,k) = \begin{bmatrix} H \\ HF \\ HF^{2n-1} \end{bmatrix} F^k[G \quad FG \quad \cdots \quad F^{2n-1}G] = \Omega_o\Omega_c \tag{14}$$

where Ω_o and Ω_c are the observability and the controllability matrices, respectively. Let $T(n,0)$ have the singular value decomposition

$$T(n,0) = V\Sigma U \tag{15}$$

where $\Sigma = \mathrm{diag}[\sigma_1,\sigma_2,\ldots,\sigma_{2n^2}]$ consists of the singular values of $T(n,0)$, $V = [v_1,v_2,\ldots,v_{2n^2}]$ consists of the right singular vectors, and $U = [u_1,u_2,\ldots,u_{2n^2}]$ consists of the left singular vectors. If $T(n,0)$ has rank N, then all of the singular values σ_i $(i=N+1,\ldots,2n^2)$ should be zero.[7] When measurement noise is present, however, there can be difficulties in determining a gap between the computed last nonzero singular value and what effectively should be considered zero. A reasonable approach to rank detection has been derived in Ref. 2 by exploiting a concise expression for the cutoff singular value σ_i:

$$\sigma_i^2 > 4n^2\sigma^2, \qquad i=1,\ldots,N \tag{16a}$$

$$\sigma_i^2 \le 4n^2\sigma^2, \qquad i=N+1,\ldots,2n^2 \tag{16b}$$

where σ is the standard deviation of the measurement noise, being white and of zero mean. According to this detected rank N of $T(n,0)$, the identification of mass, damping, and stiffness matrices can be divided into three categories.

1) $N=2n$. In this case, the motion of the dynamic system under testing is exactly described by the n second-order differential equations in Eq. (1), and the assumed dimension n of the matrices M, C, and K has correctly reflected the information of the experimental results.

2) $N<2n$. In this case, we recognize that the number of second-order equations needed to describe the motion of the system is less than n. It means that some of the responses at the n measurement points are redundant, and the dimension n of the matrices M, C, and K must be reduced to a smaller value to best fit the experimental data.

3) $N>2n$. In this case, the n second-order differential equations in Eq. (1) are not adequate to describe completely the motion of the system. Additional measurement points are required, and the dimension of M, C, and K should be enlarged to take these additional measurements into account.

We will consider case 1 first. Actually, it is not likely to meet the condition $N=2n$ at the first trial. Therefore, the requirement of model reduction for case 2 and model refinement for case 3 are of more practical importance. The latter two cases will be discussed in the next section. Now, we assume $N=2n$, and let V_1 be the matrix consisting of the first $2n$ columns of V, and U_1 be the matrix consisting the first $2n$ rows of U. Then $T(n,0)$ can be rewritten as

$$T(n,0) = V_1 \Sigma U_1 \tag{17}$$

Since $T(n,0)$ is symmetric, we must have $U_1 = V_1^T$. A minimal realization (F',G',H') satisfying the identities

$$H'F'^{i-1}G' = \phi_i \tag{18}$$

can be constructed using the following lemma.

Lemma 1.[1,7] A minimal realization (F',G',H') satisfying Eq. (18) can be expressed as

$$H' = E^T V_1 \Sigma^{1/2} \tag{19a}$$

$$G' = \Sigma^{1/2} U_1 E \tag{19b}$$

$$F' = \Sigma^{-1/2} V_1^T T(n,1) U_1^T \Sigma^{-1/2} \tag{19c}$$

where $E^T = [I_n, 0_n, \ldots, 0_n]$.

Modal parameters can be evaluated immediately after F', G', and H' have been obtained. Let λ_i, $i=1,\ldots,2n$ be the eigenvalues of F', and ψ_i, $i=1,\ldots,2n$ be the corresponding eigenvectors of F'. Then, the modal damping rates and damped natural frequencies are simply the real and imaginary parts of λ_i. $H'[\psi_1,\ldots,\psi_{2n}]$ and $[\psi_1,\ldots,\psi_{2n}]^{-1}C'$ are the mode shapes and the initial modal amplitudes, respectively. To identify the mass, damping, and stiffness matrices, we recall the relation between (F,G,H) and (M,C,K) in Eqs. (8). Since the form of the realization (F',G',H') obtained from lemma 1 may not necessarily be identical to the form of (F,G,H) in Eqs. (8), the abstraction of (M,C,K) from (F',G',H') is not so straightforward as from (F,G,H). However, we can see that (F,G,H) and (F',G',H') are similar, i.e.,

$$HF^{i-1}G = H'F'^{i-1}G' = \phi_i \tag{20}$$

and

$$\Phi(s) = H(sI-F)^{-1}G = H'(sI-F')^{-1}G' \tag{21}$$

Hence, there exists a similarity transformation P between (F',G',H') and (F,G,H). It is this similarity transformation that gives the clue to the problem of identifying M, C, and K from (F',G',H'), as can be seen from the following theorem.

Theorem 1. A similarity transformation P that satisfies

$$F = PF'P^{-1} \tag{22a}$$

$$G = PG' \tag{22b}$$

$$H = H'P^{-1} \tag{22c}$$

is obtained as

$$P = \begin{bmatrix} H' \\ H'F' \end{bmatrix} \tag{23}$$

and mass matrix M, damping matrix C, and stiffness matrix K are determined as

$$M = (H'F'G')^{-1} \tag{24}$$

$$(K \quad C) = -MH'F'^2 \begin{bmatrix} H' \\ H'F' \end{bmatrix}^{-1} \tag{25}$$

Proof. The first statement can be proved by direct substitution. Using Eq. (23) and the definition for F, G, and H, we have

$$HP = [I_{n\times n} \quad 0_{n\times n}] \begin{bmatrix} H' \\ H'F' \end{bmatrix} = H'$$

This verifies Eq. (22c). For Eq. (22b) we have

$$PG' = \begin{bmatrix} H' \\ H'F' \end{bmatrix} G' = \begin{bmatrix} 0_{n\times n} \\ H'F'G' \end{bmatrix}$$

where the identity $H'G' = 0_{n\times n}$ is from Eq. (11a). Recall $G = [0_{n\times n} \quad M^{-T}]^T$, and if we choose $M=(H'F'G')^{-1}$, then Eq. (22b) is satisfied. This leads to the expression for the mass matrix in Eq. (23). Note that under the present case $N=2n$ this expression for M is identical to the one in Eq. (12b) or Eq. (7a), which is derived from the direct expansion of $H(s)$. To verify Eq. (22a) we have, after the substitution for F from Eq. (8),

$$\begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} H' \\ H'F' \end{bmatrix} = \begin{bmatrix} H' \\ H'F' \end{bmatrix} F' \tag{26}$$

It can be seen that the first n rows are satisfied automatically, whereas the last n rows provide just enough equalities for the determination of M, C, and K as in Eq. (25). Finally, we will show that the transformation P is invertable and unique.

Let

$$T_1 = \begin{bmatrix} \phi_1 & \phi_2 & \cdots & \phi_{2n} \\ \phi_2 & \phi_3 & \cdots & \phi_{2n+1} \end{bmatrix}$$

From Eq. (14) and Eq. (15), we know

$$V_1\Sigma^{1/2} = \begin{bmatrix} H' \\ H'F' \\ \vdots \\ H'F'^{2n-1} \end{bmatrix} \tag{27}$$

Combining Eq. (27) and Eq. (17) yields

$$T_1 = P\Sigma^{1/2}U_1$$

Then, it can be seen that

$$\text{rank}(T_1) \geq \text{rank}\left(\begin{bmatrix} \phi_1 & \phi_2 \\ \phi_2 & \phi_3 \end{bmatrix}\right) = 2n$$

where we have used the fact that $\phi_1 = 0$ and $\phi_2 = M^{-1}$. On the other hand, we must have $\text{rank}(\Sigma^{1/2}U_1) = 2n$, since Σ and U_1 are all of full rank. Applying the inequality

$$\min\left[\text{rank}(P), \text{rank}(\Sigma^{1/2}U_1)\right] \geq \text{rank}(T_1) \geq 2n$$

we obtain $\text{rank}(P) \geq 2n$. Recall that P is a $2n \times 2n$ matrix, and then $\rho(P) = 2n$. It turns out the P is invertable. To show that P is unique let

$$\bar{P} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix}$$

be another similarity transformation satisfying Eqs. (22). Then from Eqs. (22c) and (22a) we have

$$\begin{bmatrix} I_{n\times n} & 0_{n\times n} \end{bmatrix} \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix} = H' \tag{28}$$

and

$$\begin{bmatrix} 0_{n\times n} & I_{n\times n} \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 F' \\ \bar{P}_2 F' \end{bmatrix} \tag{29}$$

From Eq. (28) we have $\bar{P}_1 = H'$, while from Eq. (29) we obtain $\bar{P}_2 = \bar{P}_1 F' = H'F'$. Therefore, $\bar{P} = P$, and this proves the uniqueness.

Theorem 1 provides us a novel identification technique of the mass, damping, and stiffness matrices from a minimal realization of the system under testing. As a matter of fact, if we look over Eq. (25) and consider its relations with Eqs. (14) and (17), we can see F', $H'F'$, and $H'F'^2$ are merely the first three n rows of $V_1\Sigma^{1/2}$. Hence, once the singular value decomposition of $T(n,0)$ has been obtained, M, C, and K can be determined immediately from the elements of $V_1\Sigma^{1/2}$ without actual construction of the realization (F', G', H'). The preceding result is restated in the following corollary.

Corollary. Given the Markov parameters ϕ_i of a dynamic system whose motion is adequately described by the n second-order differential equations in Eq. (1), the mass, damping, and stiffness matrices can be identified as

$$M = \phi_2^{-1} \tag{30}$$

$$[K \quad C] = -M\Omega_o(3)\begin{bmatrix} \Omega_o(1) \\ \Omega_o(2) \end{bmatrix}^{-1} \tag{31}$$

where $\Omega_o(1)$, $\Omega_o(2)$, and $\Omega_o(3)$ are the first three n rows of Ω_o or $V_1\Sigma^{1/2}$.

The preceding result allows us to estimate M, C, and K directly from the singular values and vectors of the Hankel matrix. If, however, modal parameters are also required, then it is still necessary to have a minimal realization (F', G', H').

Model Reduction and Model Refinement

In the previous section we derived the expressions for model parameters M, C, and K in terms of the singular values and singular vectors of the Hankel matrix. The dimension (n) of M, C, and K determined in this manner is equal to the number of measurement points or the number of second-order differential equations chosen to describe the motion of the dynamic system. Since this value of n is chosen before any measurement has been performed, it may have nothing to do with the real size of the model. However, after the measurement has been performed and after the ERA has been applied to the measurement data, the number of nonzero singular values (N) provides us an accuracy indicator for the order of the dynamic system under testing. It is thus warranted to feed back this important information to the identification loop and modify M, C, and K accordingly. In the previous section, we considered the special case $n = 2N$ where the assumed dimension of M, C, and K is matched with the output of ERA. Now we will consider the remaining two cases $2n > N$ and $2n < N$.

Model Reduction ($2n > N$)

In the case $2n > N$, the dimension of M, C, and K is larger than what has been indicated by ERA, which says that the number of singular values that are larger than the cutoff singular value is less than $2n$, say, $2(n-m)$ with $m \geq 1$. This, in turn, implies that a total of $n-m$ second-order differential equations is adequate to completely describe the motion of the system. Therefore, we only need the experimental data pertaining to these $n-m$ measurement points to establish the correct model.

Let $M(n)$, $C(n)$, and $K(n)$ be the parameters obtained in theorem 1 using n measurement points, while $M(n-m)$, $C(n-m)$, $K(n-m)$ be those to be determined using $n-m$ measurement points. The model reduction problem considered here is to construct the reduced-order model $M(n-m)$, $C(n-m)$, $K(n-m)$ directly from $M(n)$, $C(n)$, $K(n)$. These $n-m$ measurement points can be chosen by noting that there are only $n-m$ linear independent rows (or columns) in the impulse response matrix $h^{(n)}(t)$. There are many laboratory routines available for determining these $n-m$ linearly independent rows, for example, the row-searching method.[7] Since $h^{(n)}(t)$ is symmetric, we delete the m dependent rows and the corresponding m dependent columns from $h^{(n)}(t)$, and let the remaining matrix be denoted by $h^{(n-m)}(t)$. Without loss of generality, we partition the impulse response matrix as

$$h^{(n)}(t) = \begin{bmatrix} h^{(n-m)}(t) & h_a^{(m)}(t) \\ h_a^{(m)T}(t) & h_b^{(m)}(t) \end{bmatrix}$$

by rearranging the measurement points. Each Markov parameter is partitioned in a similar manner:

$$\phi_i(n) = \begin{bmatrix} \phi_i(n-m) & \phi_i^a(m) \\ \phi_i^{a^T}(m) & \phi_i^b(m) \end{bmatrix}$$

where $\phi_i(n-m)$, $\phi_i^a(m)$, and $\phi_i^b(m)$ are of dimensions $(n-m)\times(n-m)$, $(n-m)\times m$, and $m \times m$, respectively. Markov parameters are related to impulse response data via the formula[11]

$$\phi_k(n-m) = \sum_{i=1}^{l} \tau_{ki} h^{(n-m)}(iT)$$

$$\phi_k^a(m) = \sum_{i=1}^{l} \tau_{ki} h_a^{(m)}(iT)$$

$$\phi_k^b(m) = \sum_{i=1}^{l} \tau_{ki} h_b^{(m)}(iT)$$

where

$$[\tau_{ki}] = \begin{bmatrix} t_1 & t_1^2/2! & \cdots & t_1^l/m! \\ t_2 & t_2^2/2! & \cdots & t_2^l/m! \\ t_l & t_l^2/2! & \cdots & t_l^l/m! \end{bmatrix}^{-1}$$

with $t_1, t_2, \ldots, t_l$ being the discrete instants at which the impulse response data is recorded. From this relation, we recognize that $\phi_i(n-m)$ is, indeed, the Markov parameter of the dynamic system whose motion is described by the response at the $n-m$ spatial positions. With this observation and the use of Eq. (30) we have

$$M(n) = \begin{bmatrix} M_a(n) & M_b(n) \\ M_b(n)^T & M_c(n) \end{bmatrix} = \phi_2(n)^{-1}$$

$$= \begin{bmatrix} \phi_2(n-m) & \phi_2^a(m) \\ \phi_2^{aT}(m) & \phi_2^b(m) \end{bmatrix}^{-1} = \begin{bmatrix} M^{-1}(n-m) & \phi_2^a(m) \\ \phi_2^{aT}(m) & \phi_2^b(m) \end{bmatrix}^{-1}$$

where we have used the fact $M(n) = \phi_2(n)^{-1}$ and $M(n-m) = \phi_2(n-m)^{-1}$. Solving for $M(n-m)$ in terms of the partition matrices of $M(n)$, we have

$$M(n-m) = M_a(n) - M_b(n) M_c^{-1}(n) M_b(n)^T \tag{32}$$

where $M_a(n)$, $M_b(n)$, and $M_c(n)$ are of dimensions $(n-m) \times (n-m)$, $(n-m) \times m$, and $m \times m$, respectively.

Now, we consider the relation between $C(n)$, $K(n)$, and $C(n-m)$, $K(n-m)$. In connection with the partition of ϕ_i, $T(n,0)$ can be expressed in terms of $T(n-m,0)$ as

$$\Lambda T(n,0) \Lambda^T = \begin{bmatrix} T(n-m,0) & T_a(m,0) \\ T_a^T(m,0) & T_b(m,0) \end{bmatrix} \tag{33}$$

where Λ is a permutation matrix corresponding to the rearrangement of the columns and the rows of $T(n,0)$. Let $\alpha_{i,j}$, $i = 1,\ldots,4n, j = 1,\ldots,4n$ be the block elements of Λ. It can be seen that $\alpha_{i,2i-1} = I_{(n-m)}$, $\alpha_{2n+i,2i} = I_m$, for $1 \le i \le 2n$, and all other α_i are null elements. Let $T(n,0)$ have the singular value decomposition

$$T(n,0) = V(n) \Sigma(n) V(n)^T$$

where $\Sigma(n) = \mathrm{diag}[\sigma_1,\ldots,\sigma_{2n-2m},0,\ldots,0]$ consists of $2(n-m)$ nonzero singular values of $T(n,0)$. We define

$$\Lambda V(n) = \begin{bmatrix} V_a(n) \\ V_b(n) \end{bmatrix} \tag{34}$$

where $V_a(n)$ and $V_b(n)$ are of dimensions $2n(n-m) \times 2(n-m)$ and $2mn \times 2(n-m)$, respectively. Then from Eq. (33) we have

$$T(n-m,0) = V_a(n) \Sigma(n) V_a(n)^T \tag{35}$$

A minimal realization associated with $T(n-m,0)$ is obtained by using lemma 1:

$$H'(n-m) = E^T V_a(n) \Sigma(n)^{1/2} \tag{36a}$$

$$G'(n-m) = \Sigma(n)^{1/2} V_a(n)^T E \tag{36b}$$

$$F'(n-m) = \Sigma(n)^{-1/2} V_a(n)^T T(n-m,1) V_a(n) \Sigma(n)^{-1/2} \tag{36c}$$

where $E^T = [I_{n-m}, 0_{n-m}, \ldots, 0_{n-m}]$ and $T(n-m,1)$ is related to $T(n,1)$ in the same manner as $T(n-m,0)$ is related to $T(n,0)$ in Eq. (33). Note that Eq. (27) is also valid here, i.e.,

$$\begin{bmatrix} H'(n-m) \\ H'(n-m)F'(n-m) \\ H'(n-m)F'^2(n-m) \\ \vdots \\ H'(n-m)F'^{2n-1}(n-m) \end{bmatrix} = V_a(n) \Sigma(n)^{1/2} \tag{37}$$

The damping and stiffness matrices for the reduced-order system now can be constructed according to theorem 1:

$$[K(n-m) \quad C(n-m)] = -M(n-m) H'(n-m) \times F'^2(n-m) \begin{bmatrix} H'(n-m) \\ H'(n-m)F'(n-m) \end{bmatrix}^{-1} \tag{38}$$

where $M(n-m)$ is obtained from Eq. (32), or we can use $M(n-m) = [H'(n-m)F'(n-m)G'(n-m)]^{-1}$ instead.

Equation (38) can be further simplified to allow us to estimate $C(n-m)$ and $K(n-m)$ directly from $C(n)$ and $K(n)$. For convenience of reference, Eq. (26) is rewritten in abbreviation

$$F(n)P(n) = Q(n) \tag{39}$$

where

$$F(n) = \begin{bmatrix} 0_{n \times n} & I_{n \times n} \\ -M^{-1}(n)K(n) & -M^{-1}(n)C(n) \end{bmatrix}$$

$$P(n) = \begin{bmatrix} H'(n) \\ H'(n)F'(n) \end{bmatrix}$$

and

$$Q(n) = \begin{bmatrix} H'(n)F'(n) \\ H'(n)F'(n)^2 \end{bmatrix}$$

A similar relation exists in the reduced-order system, i.e.,

$$F(n-m)P(n-m) = Q(n-m) \tag{40}$$

Inspecting the structure of $P(i)$ and $Q(i)$ with the help of Eqs. (27) and (37), it can be seen that

$$\Lambda_1 P(n) = \begin{bmatrix} P(n-m) & \Delta P_1 \\ \Delta P_2 & \Delta P_3 \end{bmatrix}, \quad \Lambda_1 Q(n) = \begin{bmatrix} Q(n-m) & \Delta Q_1 \\ \Delta Q_2 & \Delta Q_3 \end{bmatrix}$$

where

$$\Lambda_1 = \begin{bmatrix} I_{n-m} & 0 & 0 & 0 \\ 0 & 0 & I_{n-m} & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & 0 & I_m \end{bmatrix}$$

which is, indeed, a submatrix of Λ defined in Eq. (33). Note that in the present case the rank of Hankel matrix $T(n,0)$ is $2(n-m)$, i.e., the singular values σ_i, $i = 2(n-m)+1,\ldots,2n$ are all smaller than the cutoff singular value. This implies, as we can see from Eq. (27), that ΔP_i and ΔQ_i are all of small quantity.

Multiplying both sides of Eq. (39) by Λ_1 yields

$$\left[\Lambda_1 F(n) \Lambda_1^T\right] \Lambda_1 P(n) = \Lambda_1 Q(n)$$

or, equivalently,

$$\left[\Lambda_1 F(n) \Lambda_1^T\right] \begin{bmatrix} P(n-m) & \Delta P_1 \\ \Delta P_2 & \Delta P_3 \end{bmatrix} = \begin{bmatrix} Q(n-m) & \Delta Q_1 \\ \Delta Q_2 & \Delta Q_3 \end{bmatrix} \tag{41}$$

Comparing Eq. (40) with Eq. (41) and noting that ΔP_i and ΔQ_i are small quantities yields

$$F(n-m) = L^T \Lambda_1 F(n) \Lambda_1^T L \tag{42}$$

where $L = [I_{2(n-m)\times 2(n-m)} \quad 0_{2(n-m)\times 2(n-m)}]^T$. Let

$$M^{-1}(n) = \begin{bmatrix} M_1(n) \\ M_2(n) \end{bmatrix}, \qquad K(n) = [K_1(n) \quad K_2(n)]$$

$$C(n) = [C_1(n) \quad C_2(n)]$$

where $K_1(n)$ and $C_1(n)$ are of dimenson $n \times (n-m)$, and $K_2(n)$ and $C_2(n)$ are of dimension $n \times m$. Then from Eq. (42) and the definition of $F(n)$, $F(n-m)$ and Λ_1, we have

$$M^{-1}(n-m)K(n-m) = M_1(n)K_1(n) \tag{43a}$$

and

$$M^{-1}(n-m)C(n-m) = M_1(n)C_1(n) \quad (43b)$$

Substituting $M(n-m)$ obtained from Eq. (32) into the preceding equation yields

$$K(n-m) = K_{11}(n) - M_b(n)M_c^{-1}(n)K_{21}(n) \quad (44a)$$

$$C(n-m) = C_{11}(n) - M_b(n)M_c^{-1}(n)C_{21}(n) \quad (44b)$$

where $C_1^T = [C_{11}^T \;\; C_{21}^T]^T$ and $K_1^T = [K_{11}^T \;\; K_{21}^T]^T$. The preceding results are summarized in the theorem that follows.

Theorem 2. Let $M(n)$, $C(n)$, and $K(n)$ be the mass, damping, and stiffness matrices that are estimated from the given Markov parameters ϕ_i constructed from the impulse response data at n spatial positions, and let $M(n-m)$, $C(n-m)$, and $K(n-m)$ be those obtained when the experimental data belonging to the m $(m<n)$ spatial points is removed. Then we have the following relations:

$$M(n-m) = M_a(n) - M_b(n)M_c^{-1}(n)M_b(n)^T$$

$$K(n-m) = K_{11}(n) - M_b(n)M_c^{-1}(n)K_{21}(n)$$

$$C(n-m) = C_{11}(n) - M_b(n)M_c^{-1}(n)C_{21}(n)$$

In the application of theorem 2, if m is determined according to the outputs of ERA such as singular values or degree of modal purity,[2] then the estimated parameters will reflect the true order of the system. On the other hand, if m is chosen arbitrarily, theorem 2 provides an approximation for the reduced-order system with the assumed dimension m.

Model Refinement ($2n<N$)

There are times when the data are acquired sequentially rather than in a batch, and other times when one wishes to examine the nature of the solution as more data are included to see whether some improvement in the estimated parameter continues to be made, or whether any surprises occur such as a sudden change or a persistent drift in one or more of the parameters. In short, one wishes sometimes to do a visual and experimental examination of the estimate if one or several more data points are included in the computed values of M, C, and K.

We begin with Eqs. (24) and (25) as solved for n measurement points, and consider the consequences of taking one more measurement point. We need to consider the structure of the Hankel matrix as one more datum is added. The Markov parameters can be written as

$$\phi_i(n+1) = \begin{bmatrix} \phi_i(n) & \Delta\phi_i^a \\ \Delta\phi_i^{aT} & \Delta\phi_i^b \end{bmatrix}$$

where $\Delta\phi_i^a$ and $\Delta\phi_i^b$ are of dimensions $n\times 1$ and 1×1, respectively, which are obtained from the measurement of the new test point. The singular value decomposition of the updated Hankel matrix gives

$$\Lambda T(n+1,0)\Lambda = \begin{bmatrix} T(n,0) & \Delta T^a \\ \Delta T^{aT} & \Delta T^b \end{bmatrix} = V(n+1) \times \Sigma(n+1)V^T(n+1)$$

where Λ is a permutation matrix with the block element $a_{i,2i-1} = I_{n-1}$, $a_{2n+1+i,2i} = 1$, $1\le i\le 2n+1$, and $a_{ij}=0$ elsewhere. $V(n+1)$ and $\Sigma(n+1)$ can be partitioned as

$$V(n+1) = \begin{bmatrix} V(n) & \Delta V^a \\ \Delta V^b & \Delta V^c \end{bmatrix}, \qquad \Sigma(n+1) = \begin{bmatrix} \Sigma(n) & 0 \\ 0 & \Delta\Sigma \end{bmatrix}$$

where $V(n)$ and $\Sigma(n)$ have been determined in Eq. (15). Since $V(n+1)$ must be a unitary matrix, it is not difficult to verify that

$$\Delta V^a = 0 \quad (45a)$$

$$\Delta V^b = \Delta T^{aT} V^T(n)\Sigma(n)^{-1} \quad (45b)$$

$$\Delta T^b = \Delta V^c \Delta\Sigma \Delta V^{cT} \quad (45c)$$

It turns out that the singular value decomposition of $T(n+1,0)$ reduces to that of ΔT^b. We also note that the dimensions of $T(n+1,0)$ and ΔT^b are $2n^2\times 2n^2$ and $2n\times 2n$, respectively, and then the computation time for the singular value decomposition of ΔT^b is much less than that of $T(n+1,0)$. Using Eqs. (24) and (25) and following a similar procedure for model reduction, we have the following recursive formulas for model refinement.

Theorem 3. If the impulse response of an additional point is included in the calculation of mass, damping, and stiffness matrices, we have the updated M, C, K as

$$M(n+1) = \begin{bmatrix} M_a(n+1) & M_b(n+1) \\ M_b^T(n+1) & M_c(n+1) \end{bmatrix} \quad (46)$$

where

$$M_a(n+1) = [I - M(n)\Delta\phi_2^a X \Delta\phi_2^{aT}]M(n)$$

$$M_b(n+1) = M(n)\Delta\phi_2^a(n+1)X$$

$$M_c(n+1) = -X\Delta\phi_2^b M(n)$$

$$X^{-1} = \Delta\phi_2^{aT} M(n)\Delta\phi_2^a - \Delta\phi_2^b$$

and

$$[K(n+1) \quad C(n+1)] = -M(n+1) \times \begin{bmatrix} H'(n)F'^2(n) & 0 \\ \Delta V_3^b & \Delta V_3^c \end{bmatrix} \begin{bmatrix} H'(n) & 0 \\ \Delta V_1^b & \Delta V_1^c \\ H'(n)F'(n) & 0 \\ \Delta V_2^b & \Delta V_2^c \end{bmatrix}^{-1} \quad (47)$$

where the terms with index n denote the results using n measurements, while the terms with Δ denote the quantities resulting from the response of the additional measurement point; ΔV_i^b and ΔV_i^c, $i=1,2,3$, stand for the ith row of $\Delta V^b\Sigma(n)^{1/2}$ and $\Delta V^c\Delta\Sigma_1^{1/2}$, respectively, with $\Delta\Sigma_1 = \mathrm{diag}[\sigma_{2n+1} \;\; \sigma_{2n+2}]$ consisting of the first two singular values of ΔT^b. Using these recursive formulas, the model parameters M, C, and K can be adaptively estimated if the responses of some new points have been included.

The procedures for parameter identification using ERA are described in Fig. 1. The computational steps include the following:

1) Construction of a block-Hankel matrix $T(n,0)$ using the impulse response data at n spatial positions [Eq. (13)].

2) Decomposition of $T(n,0)$ using singular value decomposition [Eq. (15)].

3) Determination of the order of the system by examining the singular values of the Hankel matrix $T(n,0)$.

4) Construction of a minimum-order realization (F',G',H') using Eqs. (19).

Let N be the number of nonzero singular values. If $N=2n$, then the order of the system is n, and the mass, damping, and stiffness matrices of the system are determined from theorem 1. If $N<2n$, the order of the system is lower than n, and M, C, and K for the reduced-order system are calculated from theorem 2. If $N>2n$, the order of the system is larger than n, and additional measurement data are required to improve the model. The updated M, C, and K are then determined from theorem 3.

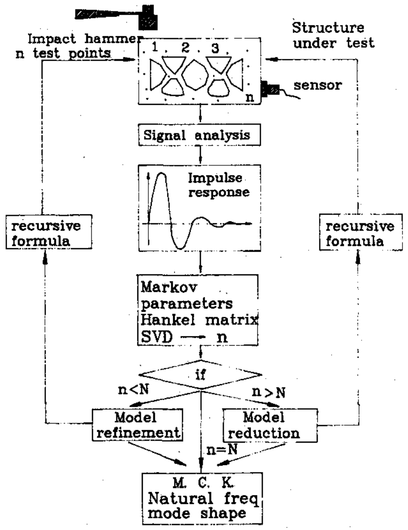

Fig. 1 Block diagram for structural parameter identification by the ERA method.

Numerical Examples

The following simple numerical example is intended to illustrate the computation procedures of the identification scheme presented here. Figure 2 shows a spring-mass damper with n degrees of freedom. The simulation data are generated from an analytical model with the parameters

$$M = \text{diag}[1 \quad 1 \quad 1]$$

$$C = \begin{bmatrix} 0.2 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 1.1 \end{bmatrix}$$

$$K = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 22 \end{bmatrix}$$

The large values for C and K are introduced intentionally to keep mass 4 nearly stationary in the model-reduction part of the example. We first assume an idealized measurement process. The impulse response matrix $h(t)$ of the three masses are recorded, and the Markov parameters ϕ_i are then determined from $h(t)$. The singular value decomposition of $T(n,0)$ defined in Eq. (13) gives

$$\sigma_1 = 1629.54, \qquad \sigma_2 = 150.600, \qquad \sigma_3 = 51.4107$$

$$\sigma_4 = 24.6528, \qquad \sigma_5 = 4.99611, \qquad \sigma_6 = 4.24143$$

$$\sigma_7 = 3.65280 \times 10^{-15}, \qquad \sigma_8 = 6.09580 \times 10^{-16}$$

Clearly, after the sixth singular value, the magnitude of the singular values is mainly due to the computer roundoff errors, and σ_7 and σ_8 can be identified with zero. Since for a spring-mass damper system each degree of freedom has two states, the simulation results show apparently that a 3 degree-of-freedom system has been tested. Using corollary 1, M, C, and K are calculated as

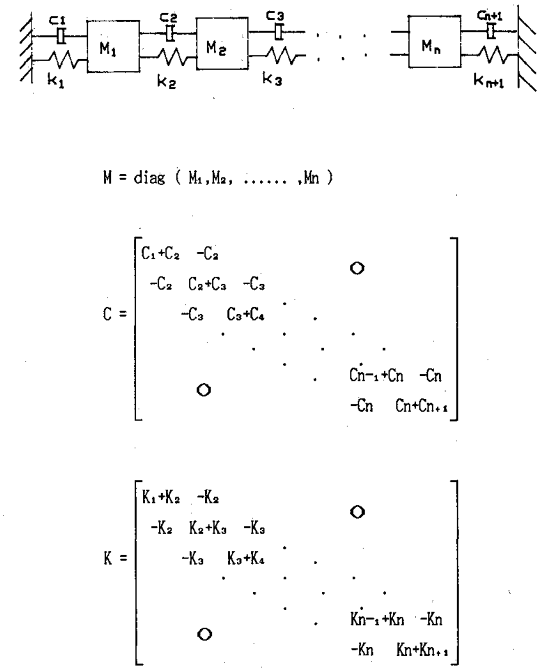

Fig. 2 An nth-order spring-mass damper system.

The eigenvalues of the matrix F' are found as

$$\lambda_{1,2} = -0.555555 \pm 4.68119i$$

$$\lambda_{3,4} = -0.146994 \pm 2.42037i$$

$$\lambda_{5,6} = -4.74509 \times 10^{-2} \pm 1.37687i$$

hence, the three damped natural frequencies in increasing order are 1.37687, 2.42037, 4.68119, and the corresponding damping rates are −0.555555, −0.146994, and -4.74509×10^{-2}. Comparing the estimated parameters with the assumed analytical model, we see that the results are exact up to the computer accuracy limit. Of course, this highly accurate result is due to the assumption that there is no noise in the measurement process.

Next we consider the case where the noise $w(t)$ with zero mean is imposed on the measurement of the impulse response data $h(t)$, i.e.,

$$h(t) = h_0(t) + \rho h_{0_{\max}} w(t) I_n$$

where $w(t)$ is a normal process with zero mean and with covariance σ_w, and ρ is the signal-to-noise ratio. Define the identification error

$$\epsilon_M = \max_{i,j}\left(\frac{|M_{ij}^0 - M_{ij}|}{M_{ij}^0}\right)$$

$$M = \begin{bmatrix} 0.100000 \times 10^1 & 0.326128 \times 10^{-15} & 0.123165 \times 10^{-15} \\ -0.277556 \times 10^{-15} & 0.100000 \times 10^1 & -0.457100 \times 10^{-15} \\ 0.706900 \times 10^{-16} & -0.336536 \times 10^{-15} & 0.100000 \times 10^1 \end{bmatrix}$$

$$C = \begin{bmatrix} 0.200000 \times 10^0 & -0.100000 \times 10^0 & 0.192433 \times 10^{-14} \\ -0.100000 \times 10^0 & 0.200000 \times 10^0 & -0.100000 \times 10^0 \\ -0.332191 \times 10^{-14} & -0.100000 \times 10^0 & 0.110000 \times 10^1 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.400000 \times 10^1 & -0.200000 \times 10^1 & 0.381639 \times 10^{-15} \\ -0.200000 \times 10^1 & 0.400000 \times 10^1 & -0.200000 \times 10^1 \\ -0.110675 \times 10^{-14} & -0.200000 \times 10^1 & 0.220000 \times 10^2 \end{bmatrix}$$

where M_{ij}^0 and M_{ij} are the elements of M for the analytical model and the estimated model, respectively. Note that ϵ_M is the maximum possible relative error occurring between the corresponding elements of M^0 and M; ϵ_K and ϵ_C are defined in a similar manner. Taking $\rho = 1\%$ and $\sigma_w = 0.2$ as an example, for the third-order system considered earlier, we have the singular values

$$\sigma_1 = 1630.86, \qquad \sigma_2 = 150.514, \qquad \sigma_3 = 52.9899$$

$$\sigma_4 = 25.1713, \qquad \sigma_5 = 5.00190, \qquad \sigma_6 = 4.99038$$

$$\sigma_7 = 0.330201, \qquad \sigma_8 = 0.102416$$

Though the impulse response data is now coupled with the measurement noise, it is evident that the first six singular values originate from the system, whereas the last two are due to noise effects. This trend also can be seen from Eqs. (16). In this case, we have the cutoff singular value $2n\sigma_w = 1.2$. Since $\sigma_6 > 1.2$ and $\sigma_7 < 1.2$, we thus arrive at the same conclusion. The estimation error in damped natural frequency is found as

$$\Delta\omega_1 = 2.69 \times 10^{-4}, \qquad \Delta\omega_2 = 2.35 \times 10^{-2}$$

$$\Delta\omega_3 = 4.27 \times 10^{-1}$$

and the identification error in M, C, K is

$$\epsilon_M = 7\%, \qquad \epsilon_C = 15\%, \qquad \epsilon_K = 2\%$$

For a still higher value of ρ, the singular values originating from the system or from the noise are nearly indistinguishable; in such a case, Eqs. (16) may be helpful to determine the real order of the system. Figure 3 shows the variation of the identification error ϵ_M with respect to the degrees of freedom for various signal-to-noise ratios. As mentioned in the last section, the determination of ϕ_i from $h(t)$ needs the inverse of the matrix

$$\begin{bmatrix} t_1 & t_1^2/2! & \ldots & t_1^l/l! \\ t_2 & t_2^2/2! & \ldots & t_2^l/l! \\ t_l & t_l^2/2! & \ldots & t_l^l/l! \end{bmatrix}$$

where $t_1, t_2, \ldots, t_l$ are the discrete instants at which the impulse response data are recorded. When the value of l becomes larger and larger, this matrix becomes ill-conditioned, and the estimation accuracy is deteriorated rapidly in the presence of noisy data, as can be seen in Fig. 3 for the cases $\rho = 1\%$ and $\rho = 0.5\%$. If ρ is small, however, the identification error is nearly independent of the degree of freedom of the system (see the curve associated with $\rho = 0.1\%$ in Fig. 3). In the limiting case where ρ approaches to 0, the identification error is a constant value very close to zero as we have seen in the preceding noise-free example.

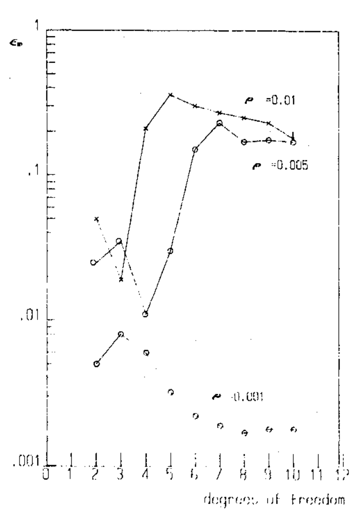

Fig. 3 Identification error ϵ_M vs degrees of freedom for various noise ratio.

For the model-reduction part, a better example will be the problem of approximating a system with infinite degree of freedom by a finite degree-of-freedom system to fully reflect the practical significance of the above results. However, to continue the discussion of the preceding identification part, we still consider the simple system shown in Fig. 2. The impulse responses at the four masses are recorded. In a noise-free measurement process, the estimated parameters are identical to the analytical parameters that follow.

$$M(4) = \text{diag}[1 \quad 1 \quad 1 \quad 1]$$

$$C(4) = \begin{bmatrix} 0.2 & -0.1 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 \\ 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & -0.1 & 1.1 \end{bmatrix}$$

$$K(4) = \begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 4 & -2 & 0 \\ 0 & -2 & 4 & -2 \\ 0 & 0 & -2 & 22 \end{bmatrix}$$

We now wish to obtain a reduced-order model of the system by removing the measurement data of the right mass. Note that this model-reduction problem is quite artificial since the output of ERA indicates that there are eight nonzero singular values, and there is no evidence that the system can be well approximated by a lower-order model. However, it is interesting to see what will happen if we artificially remove the measurement data of the right mass. According to theorem 2 we have

$$M(3) = M_a(4) - M_b(4)M_c^{-1}(4)M_b(4)^T = \text{diag}[1 \quad 1 \quad 1]$$

$$C(3) = C_{11}(4) - M_b(4)M_c^{-1}(4)C_{21}(4) = \begin{bmatrix} 0.2 & -0.1 & 0 \\ -0.1 & 0.2 & -0.1 \\ 0 & -0.1 & 0.2 \end{bmatrix}$$

$$K(3) = K_{11}(4) - M_b(4)M_c^{-1}(4)K_{21}(4) = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix}$$

These are just the mass, damping, and stiffness matrices for the 3 degree-of-freedom system containing the first three masses of Fig. 2. This result is consistent with the previous assumption that measurement data pertaining to the motion of the right mass is ignored in the construction of the Hankel matrix. From this observation we know that the model-reduction algorithm proposed here will give the exact reduced-order model of order $n - m$, if the system is truly of an order $n - m$, while being described by the motions of n measurement points of the system.

Conclusions

Two developments are given in this paper. First, a novel scheme is proposed to identify the model parameters of vibrating structures. The eigensystem realization algorithm (ERA) is exploited and extended to identify the physical properties of mass, damping, and stiffness matrices. No assumption regarding the nature of damping is made other than it is of the viscous type. A remarkable property of this identification scheme is its closed-loop nature, which feeds back the output of ERA to modify the elements and the dimensions of these matrices progressively until the model can completely fit the experimental data.

Second, model reduction and model-refinement algorithms are developed to incorporate the preceding identification technique to form a closed-loop framework wherein the ERA method provides the basis for a rational choice of model size. From the computational standpoint, the algorithm is numerically stable since only simple operations (singular value decomposition) are needed. Numerical examples are considered, and the effectiveness and the validity of the proposed algorithm are shown.

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